

FLAT CONFORMAL STRUCTURES ON 3-MANIFOLDS, I: UNIFORMIZATION OF CLOSED SEIFERT MANIFOLDS

MICHAEL KAPOVICH

Abstract

This is the first in a series of papers where we prove an existence theorem for flat conformal structures on finite-sheeted coverings over a wide class of Haken manifolds

Introduction

A *flat conformal structure* on a manifold M (of dimension $n > 2$) is a maximal atlas

$$K = \{(U_i, \varphi_i), \varphi_i: U_i \rightarrow V_i \subset \mathbb{S}^n, i \in I\}$$

with conformal transition maps $\varphi_i \circ \varphi_j^{-1}$. From more classical point of view a flat conformal structure (FCS) is a conformal class of conformally Euclidean Riemannian metrics on M . This definition is equivalent to the former one (see [34], [39], e.g.). The best-known way to construct FCS is by *uniformization*: If a Kleinian group Γ acts freely and discontinuously on a domain $D \subset \mathbb{S}^n$, then a flat conformal structure K_Γ naturally arises on the factor manifold $M = D/\Gamma$. For this structure K_Γ the covering $p: D \rightarrow M$ is a conformal map. Such structures are called *uniformizable*, and Γ is called the *uniformizing* group. Five 3-dimensional geometries [56] are conformally Euclidean: \mathbb{S}^3 , \mathbb{E}^3 , $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, \mathbb{H}^3 .

The abundance of FCS in the dimension 3 is provided by the following well-known result of Thurston.

Theorem H [51], [53], [57], [58], [59]. *Let M be a closed atoroidal Haken 3-manifold. Then M admits a hyperbolic structure.*

According to Kulkarni [38] FCS exists on connected sum of conformally flat manifolds. On the other hand, Goldman [9] has shown that any closed 3-manifold M , modeled on Sol- or Nil-geometry, does not admit a flat conformal structure.

This paper is the first in a series of three establishing existence of FCS on a wider class of 3-manifolds than provided by the theorems of Thurston and Kulkarni; namely,

Theorem 5.1. *Let M be a closed Haken 3-manifold with unsolvable fundamental group such that the canonical composition of M from hyperbolic and Seifert components does not include gluing hyperbolic manifolds with hyperbolic or Euclidean ones. Then some finite-sheeted covering of M admits a uniformizable flat conformal structure.*

Remark 1. Remind that a Euclidean manifold (in the sense of [56]) is a compact manifold N such that $\text{int}(N)$ admits a complete Euclidean structure (i.e., a flat Riemannian metric). There are only two Euclidean 3-manifolds with boundary, all of them are covered by $\mathbb{S}^1 \times \mathbb{S}^1 \times [0, 1]$. Therefore, if a closed 3-manifold M is obtained by gluing hyperbolic and Euclidean components H and E , then a 2-sheeted covering of M is obtained by gluing two copies of the manifold H .

The first Russian version of Theorem 5.1 was published in [23], where the condition on hyperbolic-Euclidean gluing was mistakenly dropped. A corrected exposition (in Russian) may be found in [25], where we consider the case of graphmanifolds.

Theorem 5.1 combined with the Kulkarni's result on conformal connected sum (see above) makes the following conjecture [23] probable.

Conjecture 1. *Let M be a closed 3-manifold satisfying the Thurston Geometrization Conjecture [57], i.e., M is the result of toroidal gluing and connected sum of manifolds possessing geometric structures. Suppose also that the decomposition of M into connected sum of prime components does not include Sol- or Nil-manifolds. Then some finite-sheeted covering of M admits an uniformizable flat conformal structure (see [23]).*

For Seifert manifolds we can obtain a more precise result than given by Theorem 5.1. Namely, in the present paper we prove

Theorem 2.1. *Let $S(g, e)$ be a total space of a circle bundle over a closed orientable surface S_g of a genus g having Euler number $e \in \mathbb{Z}$ such that $0 < e \leq (g-1)/11$. Then $S(g, e)$ admits a uniformizable FCS.*

An analogous result was independently obtained in the joint work of Gromov, Lawson, and Thurston [14] (see also [36], [37] for further discussion). Later in [28] it was shown that the condition $e \leq \exp(8000g/\mu)$ is necessary for existence of uniformizable FCS on $S(g, e)$, where μ is the Margulis constant for the four-dimensional hyperbolic space of the curvature $= -1$.

If $e = 0$, then a flat conformal structure on $S(g, e)$ always exists, but for $e \neq 0$, $g = 1$ the manifold $S(g, e)$ does not admit any FCS, since such manifolds are always Nil [9].

Limit sets of groups $H(g, e)$ uniformizing $S(g, e)$ are tame unknotted topological circles in (Corollary 2.3). Such groups are called *pseudofuchsian*. In Corollary 2.2 we use pseudofuchsian groups $H(g, e)$ to construct examples of uniformly quasiconformal actions not topologically equivalent to conformal actions. The examples of such kind were constructed also by Tukia [62], Freedman and Skora [7], and Martin [45] (see §2.9). Furthermore, Isachenko [18], applying Theorems 5.1 and 6.1, constructed an example of discrete uniformly quasiconformal action on \mathbb{S}^3 of a group Γ which is not isomorphic to any subgroup of $\text{Isom}(\mathbb{H}^4)$ (and does not even admit nonsolvable representations).

Theorems 2.1 and 5.1 present other interesting examples of pathological properties of flat conformal structures: disconnectedness of the moduli space $C(M)$ of all FCS on the manifold $M = S(g, e)$. Namely, we have

Theorem 2.2. *Let M be a manifold $S(g, e)$, $e \neq 0$, and let $C(M)$ be the moduli space of flat conformal structures on the manifold M . Then $C(M)$ consists of at least $[(g-1)/(11e)]$ connected components.*

Furthermore, for each $n \geq 5$ there are examples of compact n -dimensional manifolds M^n such that the space $C(M^n)$ consists of infinitely many connected components (see [29]).

In [31, Theorem 6.1] we construct an example of closed orientable 3-manifold M which does not admit any FCS but has conformally flat finite-sheeted covering M_0 . The manifold M is obtained by gluing two boundary components of some Seifert manifold. This is the first example of orientable 3-manifold which does not admit any FCS but has unsolvable fundamental group.

It is interesting to compare this result with Thurston's Realization Conjecture.

Conjecture 2. (See [52].) Let N be a compact manifold modeled on some of eight three-dimensional geometries (X, G) . Let F be a finite group acting smoothly on N . Then this action of F is isometric in some (X, G) -structure on N .

This conjecture was proven in many cases (see [52], [33]).

Consider now the deck-transformation group F of the covering $M_0 \rightarrow M$ constructed in Theorem 6.1 of [31]. The group F is finite and acts smoothly on the manifold M_0 . Then Theorem 6.1 states that the action of F is not conformal in any flat conformal structure on M_0 . So the naive analog of Thurston's Realization Conjecture is not valid for the flat conformal geometry (which is not a geometry in Thurston's sense!).

It should be mentioned that the class of 3-manifolds possessing FCS is wider than that discussed in Theorem 5.1. First we can use the conformal

connected sum. Furthermore, let M_1, M_2 be compact 3-manifolds with hyperbolic interiors. Then ∂M_j admits a canonical conformal structure ($j = 1, 2$). Suppose that $f: \partial M_1 \rightarrow \partial M_2$ is isotopic to a conformal map of the boundary tori. Then the conformal structures on M_1, M_2 are glued by f to uniformizable FCS on $M = M_1 \cup_f M_2$. This fact can be deduced directly from the Maskit Combination Theorem; see also [38], [39], [14].

However such gluing is impossible if the sewing map is not isotopic to a conformal one. Hence, to find a FCS on $M_1 \cup_f M_2$ in the general case we have to deform hyperbolic structures on M_j in the space of all FCS so that for new FCS $C_{j\tau}$ the holonomy homomorphisms restricted to ∂M_j are discrete and faithful representations $\rho_{\partial j\tau}$, which are not conformally conjugate to the holonomy of the initial structures. For some hyperbolic manifold such deformation is impossible (see [27], where we consider the case of 2-bridge knots' complements).

Remark 2. The infinitesimal variation of $\rho_{\partial j\tau}$ is always trivial. However there are examples of nontrivial local deformation of $\rho_{\partial j\tau}$ [27].

One can try to find an FCS K on M looking at deformations of hyperbolic structures on M_j such that the holonomy representations $\rho_{j\tau}$ have invariant round spheres S_j^2 . Such deformations correspond to the hyperbolic Dehn surgery [60, §5] and images of $\rho_{\partial\tau}$ contain loxodromic elements. However this implies that $S_1^2 = S_2^2$. Therefore, the holonomy group of K is conjugate to a subgroup of $SO(3, 1)$ which follows that M is hyperbolic itself (see [24]). Such conclusion is definitely wrong and this naive approach fails.

However, Theorem 6.1 suggests looking for such deformations of flat conformal structures on some finite-sheeted coverings over M_j .

Conjecture 3 [23]. Let M be a closed hyperbolic 3-manifold, and let $\rho: \pi_1 M \rightarrow SO(3, 1)$ be its holonomy representation. Then $[\rho]$ is not locally rigid in $\text{Hom}(\pi_1 M, SO(4, 1))/SO(4, 1)$ iff M contains an incompressible surface which is not a virtual fiber in a fiber bundle over S^1 .

Remark 3. The existence of an incompressible surface above implies that FCS on M is not unique [10].

If this conjecture is true, then it relates three different phenomena which appear only for finite-sheeted coverings over 3-manifolds: (1) existence of FCS, (2) existence of incompressible surfaces that are not virtual fibers (Waldhausen-Gabai Conjecture), (3) nonrigidity of holonomy representations $[\rho]$ above in $\text{Hom}(\pi_1 M, SO(4, 1))/SO(4, 1)$. Conjecture 3 was proved in [27] in several cases (see [27] for further discussion).

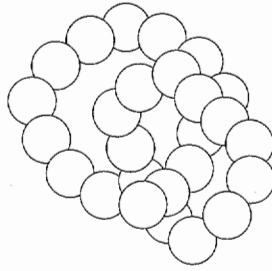


FIGURE 1

In this paper we avoid these very complicated nonrigidity problems using the abundance of deformations of flat conformal structures on Seifert manifolds.

Manifolds combined from hyperbolic and Seifert components can be uniformized in essentially different ways. One way is presented by the proof of Theorem 5.1 (see Example 2). Another way can be found in [14] where there are constructed discrete groups G whose limit sets are wild knots, and G uniformizes $(\mathbb{S}^3 \setminus K) \cup (\Sigma \times \mathbb{S}^1)$ where K is the tubular neighborhood of an arbitrary nontrivial knot, and Σ is a compact surface with one boundary component and sufficiently large genus. Moreover, some Haken 3-manifolds combined from hyperbolic and Seifert components can be uniformized by discrete groups whose limit sets are wild Cantor sets [11], [13], [49]. Even hyperbolic 3-manifolds can be uniformized in a very pathological way [2].

Idea of the proof of Theorem 5.1. We present here two examples which explain forthcoming constructions and illustrate arising difficulties.

Example 1. Let $Z_j = \Sigma_j \times \mathbb{S}^1$, $j = 1, 2$, where Σ_j is a surface of genus $g_j \neq 0$ and has connected boundary. The decomposition of Z_j into the direct product introduces a natural meridian-longitude basis in $\pi_1(\partial Z_j)$. Suppose that the manifold M is obtained by gluing Z_j via a homeomorphism $f: \partial Z_1 \rightarrow \partial Z_2$ which is defined (in the natural bases) by a matrix $A \in GL_2(\mathbb{Z})$ with $a_{21} = 1$, $\det(A) = -1$. If the numbers g_k are sufficiently large with respect to $|a_{jj}|$, then there exist groups $H_1 = H(g_1, |a_{22}|)$, $H_2 = H(g_2, |a_{11}|)$ uniformizing the manifolds $S(g_1, |a_{22}|)$, $S(g_2, |a_{11}|)$ (Theorem 2.1).

We can find fundamental domains Φ_j for H_j , which are homeomorphic to unknotted solid tori so that the complements of Φ_1, Φ_2 define a simplest link in link of index 1 in \mathbb{S}^3 (Figure 1). Let T be a two-dimensional torus in the interior of $\Phi = \text{cl}(\Phi_1 \cap \Phi_2)$ which is isotopic to

boundary components; let T_j^0 be the component of $S^3 \setminus T$ which lies in Φ_j . Applying the Klein-Maskit Combination Theorem (Theorem 1.1) we see that Φ is a fundamental domain for the group

$$G = H(g_1, |a_{22}|) * H(g_2, |a_{11}|).$$

The factor manifold Φ/G can be obtained by gluing faces of Φ by generators of H_1, H_2 ; thus

$$\Phi/G = \text{cl}(\Phi_1 \setminus T_1^0)/H_1 \cup_T \text{cl}(\Phi_2 \setminus T_2^0)/H_2.$$

However every component $\text{cl}(\Phi_j \setminus T_j^0)/H_j$ is homeomorphic to $Z_j = \Sigma_j \times S^1$ ($j = 1, 2$). Then $\Phi/G = Z_1 \cup_T Z_2$. Looking at the gluing map f' here we see that the fiber of Z_1 is identified with a loop on ∂Z_2 , which goes once around $\partial \Sigma_2$ and a_{11} times along the fiber of Z_2 (after appropriate choice of the orientation). The same consideration for the fiber of Z_2 shows that the matrix of the gluing map f' is

$$A' = \begin{pmatrix} \pm a_{11} & 1 - (\pm a_{22} \pm a_{11}) \\ 1 & \mp a_{22} \end{pmatrix}.$$

More careful choice of H_j (we can conjugate them by orientation reversing Moebius transformations) gives the manifold Φ/G homeomorphic to M . However it is impossible to avoid the condition $|a_{21}| = 1$ (for the circumscribed construction of the group G). Proving Theorem 5.1 we find a finite-sheeted covering over M such that the corresponding coefficients a_{21} are equal to 1 for every pair of adjacent Seifert components.

Example 2. Let G_1 be a torsion-free discrete subgroup of $PSL(2, \mathbb{C})$, and $p: \mathbb{H}^3 \rightarrow \mathbb{H}^3/G_1 = M_1$ be the universal covering; the manifold M_1 is compact and contains a simple closed geodesic γ . Suppose that some component $\tilde{\gamma} \subset p^{-1}(\gamma)$ has the hyperbolic stabilizer $\langle g \rangle$ in G_1 , i.e., g preserves a hyperbolic half-plane P_γ in \mathbb{H}^3 with the boundary $\tilde{\gamma}$. Then for some $\varepsilon > 0$ the geodesic γ has an open ε -neighborhood $U_\varepsilon(\gamma)$ which is homeomorphic to the solid torus. It is not hard to notice that the interior of the manifold

$$M_1^* = M_1 \setminus U_\varepsilon(\gamma)$$

is hyperbolic [32]. Consider the unit ball $\subset \mathbb{R}^3$ as a model for \mathbb{H}^3 ; we can take P_γ to be contained in the Euclidean disc Δ with the boundary circle $C \supset \tilde{\gamma}$.

Consider Δ as a model for \mathbb{H}^2 . Let Σ^0 be a hyperbolic surface with infinite area, genus r , and one ideal boundary component. Let $\Sigma_c \subset \Sigma^0$

be the Nielsen core, i.e., the intersection of all compact convex subsurfaces homotopy equivalent to Σ^0 . Assume that

(i)
$$\text{length}(\gamma) = \text{length}(\partial\Sigma_c),$$

(ii)
$$\arccos(1/\cosh(\delta)) = \arcsin(1/\cosh(\varepsilon)),$$

and the δ -neighborhood $U_\delta(\partial\Sigma_c)$ of $\partial\Sigma_c$ is homeomorphic to the annulus.

Put $\Sigma = \Sigma_c \setminus U_\delta(\partial\Sigma_c)$. We can choose a uniformization $\Delta \supset \tilde{\Sigma} \rightarrow \Sigma$ such that $\tilde{\Sigma}$ is contained in $U_\varepsilon(\gamma) \cap P_\gamma$ and $\partial\tilde{\Sigma} \cap \partial U_\varepsilon(\gamma) = \partial U_\varepsilon(\gamma) \cap P_\gamma$ (condition (ii)).

Remark 4. The geodesic $\partial\Sigma_c \subset \Sigma^0$ is covered by $\Delta \cap \partial_\infty \mathbb{H}^3$.

Let $\Gamma \subset \text{Isom}(\mathbb{H}^2)$ be the uniformizing group, $\tilde{\Sigma}/\Gamma$ is Σ . Extend Γ to the group G_2 of conformal transformations of S^3 . Then we have:

- (1) the circle C is invariant under G_2 ,
- (2) $\langle g \rangle \subset [G_2, G_2]$ corresponds to $\pi_1(\partial\Sigma_c) \subset \pi_1(\Sigma_c)$ (due to condition (i)).

If S^1 is the circle with any homogeneous Riemannian metric, then the Riemannian structure of direct product on $\Sigma \times S^1$ is conformally flat. We can choose the length of S^1 so that $\Sigma \times S^1$ is conformally equivalent to Ω/G_2 , where the domain $\Omega \setminus C$ can be obtained by rotation of $\tilde{\Sigma}$ around C . $\Omega \cap C$ is the intersection of the discontinuity domain of G_2 with $\tilde{\gamma}$. Then the group G generated by G_1, G_2 uniformizes a manifold M which is obtained by gluing M_1^* and $\Sigma \times S^1$ along the boundary tori. Conditions (i), (ii) guarantee that ∂M_1^* and $\partial(\Sigma \times S^1)$ are Moebius equivalent.

However only few sewings may be realized in such way, and the hyperbolicity of g is the vry restrictive condition. That is why we have to waive utilizing groups G_2 with invariant circles. Instead we use discrete groups that will be constructed in [30, Theorem 4.1] after small deformations of pseudofuchsian groups. These groups provide the first type of building blocks for the proof of Theorem 5.1; they uniformize finite-sheeted coverings of Seifert components in the canonical decomposition of M .

The second type of building blocks is a class of "hyperbolic" groups. These groups uniformize incomplete hyperbolic structures on interiors of hyperbolic components of the canonical splitting of M . An example of such uniformization is given by the group G_1 in Example 1 (incomplete hyperbolic manifold is $M_1 \setminus \gamma$). The "hyperbolic" groups will be obtained by small deformations of nonuniform lattices in $PSL_2(\mathbb{C})$, i.e., confinite discrete subgroups of $PSL_2(\mathbb{C})$. The main problem is to find small

deformations of lattices and pseudofuchsian groups such that a conformal gluing of uniformized hyperbolic and Seifert manifolds is possible.

For this purpose we choose deformations of lattices such that their parabolic subgroups $\mathbb{Z} \oplus \mathbb{Z}$ become $\mathbb{Z} \oplus \mathbb{Z}_n$, generated by loxodromic and elliptic transformations, [30, §3]. At the same time cyclic parabolic subgroups of pseudofuchsian groups become loxodromic ones, which are conjugated to subgroups of the corresponding $\mathbb{Z} \oplus \mathbb{Z}_n$, as given in [30]. The elliptic elements above disappear after transition to finite-index subgroups [30]. In [30, §3] we state also some auxiliary results concerning constructions of some pseudofuchsian groups and deformation problems for lattices and free Kleinian groups. In [31] we present a direct construction of a Kleinian group uniformizing a finite-sheeted covering of M . The main tool here is Klein-Maskit Combination Theorems and some results of Hempel, McCullough, and Miller related to the residual finiteness property of 3-manifold groups. These results together with some basic facts about Kleinian groups and flat conformal structures are collected in §1.

1. Definitions and some basic facts of the theory of Kleinian groups and related topics

1.1. Let \mathbf{Mob}_n be the group of all orientation-preserving Moebius transformations of the n -sphere $\mathbb{S}^n = \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. This group is isomorphic to connected component of 1 in the Lorentz group $SO(n+1, 1)$. The fixed-point set of $\gamma \in \mathbf{Mob}_n$ is denoted by $\text{Fix}(\gamma) = \{x \in \mathbb{S}^n : \gamma(x) = x\}$. For a group $\Gamma \subset \mathbf{Mob}_n$ the *discontinuity set* $R(\Gamma)$ is equal to

$\{x \in \mathbb{S}^n : \text{the point } x \text{ possesses a neighborhood } U(x) \text{ such that the intersection } U(x) \cap \gamma(U(x)) \text{ is empty for all but finite elements } \gamma \in \Gamma\}$.

The complement $\mathbb{S}^n \setminus R(\Gamma) = L(\Gamma)$ is the *limit set* of the group Γ . A subgroup $\Gamma \subset \mathbf{Mob}_n$ is said to be *Kleinian* if $R(\Gamma)$ is not empty. So, Kleinian groups are discontinuous groups of Moebius transformations; we do not require $L(\Gamma)$ to be infinite.

Any Moebius transformation $\gamma \in \mathbf{Mob}_n$ may be canonically extended to an element $\tilde{\gamma} \in \mathbf{Mob}_{n+1}$; this extension agrees with the embedding $SO(n+1, 1) \subset SO(n+2, 1)$. The element $\tilde{\gamma}$ has a closed invariant ball $\mathbb{B}^{n+1} \subset \mathbb{S}^{n+1}$ with the boundary \mathbb{S}^n ; the interior of \mathbb{B}^{n+1} plays the role of a model for the hyperbolic space \mathbb{H}^{n+1} .

An element γ is said to be *loxodromic* if $\text{Fix}(\tilde{\gamma}) \cap \mathbb{B}^{n+1} = \{p \neq q\} \subset \mathbb{S}^n$. An element γ is said to be *parabolic* if $\text{Fix}(\tilde{\gamma}) \cap \mathbb{B}^{n+1} = \{p\} \subset \mathbb{S}^n$; and γ

is said to be elliptic in either case: $\text{Fix}(\tilde{\gamma}) \cap \text{int } \mathbb{B}^{n+1} \neq \emptyset$.

If a loxodromic element γ is conjugate in \mathbf{Mob}_n to a homothety $q: x \rightarrow kx, x \in \mathbb{R}^n$, then γ is said to be a *hyperbolic* element. A subgroup $\Gamma \subset \mathbf{Mob}_n$ is said to be *loxodromic* (or *elliptic* or *parabolic*) if every element $\gamma \in \Gamma \setminus \{1\}$ is loxodromic (or *elliptic* or *parabolic* respectively).

For a closed connected hypersurface S in \mathbb{R}^n the compact component $\text{int}(S)$ of $\mathbb{R}^n \setminus S$ is called the *interior* of this hypersurface. Analogously,

$$\text{ext}(S) = \mathbb{S}^n \setminus \text{cl}(\text{int}(S))$$

is called the *exterior* of it.

Let $\Sigma \subset \mathbb{S}^3$ be a Euclidean sphere of codimension 1. *Spherical polygon* on Σ is a closed domain bounded by a finite collection of disjoint simple loops so that each loop is a finite union of circular arcs.

The *standard annulus* is the subset of the Euclidean plane: $A_R = \{z \in \mathbb{C}: 1 \leq |z| \leq R\}$ where $1 < R < \infty$.

A *fundamental set* for the Kleinian group G is a subset Φ of $R(G)$ such that the orbit $G \cdot \Phi$ coincides with $R(G)$ and $g(\Phi) \cap \Phi = \emptyset$ for any $g \in G \setminus \{1\}$.

If $\text{cl}(\Phi)$ is bounded by a collection of spherical polygons, then Φ is called a *fundamental polyhedron* for G . In this case these spherical polygons are called to be *faces* of Φ .

For a group G its commutator subgroup is denoted by $[G, G]$.

Let $h \in \mathbf{Mob}_n$ be a loxodromic element; then the *axis* A_h of h is the geodesic in \mathbb{H}^{n+1} which joins the fixed points of h .

Let h be a loxodromic transformation of \mathbb{S}^3 , and l be any h -invariant open arc of circle \mathcal{L} that passes through $\text{Fix}(h)$.

Definition 1. The pair $(h, l) = \vec{h}$ is called a *directed loxodromic transformation*. Two directed transformations \vec{h}_1, \vec{h}_2 are said to be *conjugate* if there exists a transformation $f \in \mathbf{Mob}_3$ such that (1) $f(h_1)f^{-1} = h_2$ and (2) $f(l_2) = \mathcal{L}_1 \setminus \text{cl}(l_1)$.

Assume that the complex plane is included in \mathbb{R}^3 in the standard way: $\mathbb{C} = \{(x_1, x_2, 0), x_1 + x_2i \in \mathbb{C}\}$. Then a loxodromic transformation h is conjugate in \mathbf{Mob}_3 to an element h^* preserving \mathbb{C} , $h^*: z \mapsto k(h) \cdot z, z \in \mathbb{C}, k(h) \in \mathbb{C}^*$. The complex number $k(h)$ is independent of choice of h^* up to the conjugation $k(h) \mapsto \overline{k(h)}$ and the inversion $k(h) \mapsto (k(h))^{-1}$; we shall suppose that $\text{Im}(k(h)) \geq 0$ and $|k(h)| \leq 1$.

Definition 2. The complex number $k(h)$ is the *complex coefficient* of the loxodromic transformation h .

Let (M, d) be a metric space, $X \subset M$, $Y \subset M$. Then we put

$$d(X, Y) = \inf\{d(x, y), (x, y) \in X \times Y\},$$

and $\text{dist}(X, Y) = \sup\{\inf\{d(x, y), y \in Y\}, x \in X\}$.

Let $g \in \mathbf{Mob}_n$ be an element such that $g(\infty) \neq \infty$. Then the isometric sphere of the element g is the set $I(g) = \{x \in \mathbb{R}^n, \det(Dg(x)) = 1\}$, where $Dg(x)$ is the Jacobian matrix for the map g . Let G be a Kleinian group such that $\infty \in R(G)$. Then the set

$$I(G) = \bigcap_{g \in G - \{1\}} \text{ext}(I(g))$$

is called the *isometric fundamental polyhedron of the group G* .

If $g \in \mathbf{Mob}_n$ is arbitrary element, then $I(g^{-1}) = g(I(g))$. Let I be a round sphere in \mathbb{R}^n with center at the point O ; then P is a Euclidean hyperplane in \mathbb{R}^n such that $I \cap P = \emptyset$. Consider the Moebius transformation $g = R_P \circ J_I: \mathbb{S}^n \rightarrow \mathbb{S}^n$ where R_P is the reflection in P , and J_I is the inversion in I . Then g is a hyperbolic transformation. Suppose that ϕ is a nontrivial rotation with center at $R_P(O)$. Then $\phi \circ g$ is a loxodromic transformation which is not hyperbolic. To obtain a parabolic transformation consider the case where P is tangent to I and $\phi(P \cap I) = P \cap I$. Then $\phi \circ R_P \circ J_I$ is parabolic. It can be shown that in all cases above $I(\langle G \rangle) = \text{ext}(I(g^{-1})) \cap \text{ext}(I(g))$.

1.2. Combination theorems. The definitions and statements of this section are rather long and messy. Nice illustrative examples can be found in [42], [46]. The main idea here is the following: given two Kleinian groups $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}_n$ we are to find some conditions so that (a) the group $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ generated by the elements of Γ_1 and Γ_2 is also Kleinian and (b) the factor manifold $R(\Gamma)/\Gamma$ can be obtained by some "cut and paste operation" with the manifolds $R(\Gamma_i)/\Gamma_i$.

Definition 3. Let J be a subgroup of a group $G \subset \mathbf{Mob}_n$, and B be a subset of \mathbb{S}^3 . Then B is said to be *precisely invariant under J in the group G* if

- (1) $J(B) = B$ and
- (2) for any $g \in G \setminus J$ we have $g(B) \cap B = \emptyset$.

Definition 4. Let J be a cyclic loxodromic or trivial subgroup of $G \subset \mathbf{Mob}_3$. Then a compact manifold B , which is precisely invariant under $J \subset G$, is called a (G, J) -*block* if $B \cap R(G) = B \cap R(J)$.

Examples. Suppose that J is a trivial group, and $B \subset R(G)$ is contained inside of a fundamental domain of G . Then B is a (G, J) -*block*.

Next, let $B = \text{cl } U_\varepsilon(\gamma)$, and $G = G_1$ be the group as in Example 2; then $J = \langle \gamma \rangle$. Thus B is a (G, J) -block. These are two basic examples that we need.

Theorem 1.1 (First Maskit Combination Theorem). *Let J be a cyclic loxodromic or trivial subgroup of discrete groups $G_1, G_2 \subset \mathbf{Mob}_3$. Assume that $J \neq G_1, J \neq G_2$ and there is a closed embedded surface W dividing S^3 into two compact submanifolds B_1, B_2 , where B_m is a (J, G_m) -block, $m = 1, 2$. Let D_m be a fundamental set for G_m such that*

- (1) $D_m \cap B_m$ is a fundamental set for action of J in B_m ,
- (2) $D_1 \cap W = D_2 \cap W$,
- (3) $D_m \cap B_{3-m}$ has nonempty interior, $m = 1, 2$.

Set $D = (D_1 \cap B_2) \cup (D_2 \cap B_1)$ and $G = \langle G_1, G_2 \rangle$. Then the following statements hold.

- (i) $G \cong G_1 *_J G_2$ —free product with amalgamation over J .
- (ii) The group G is Kleinian.
- (iii) D is a fundamental set for G .

Let Q_m be the union of the G_m -translates of $\text{int}(B_m)$, and let R_m be the complement of Q_m . Then

- (iv) $R(G)/G = (R_1 \cap R(G_1))/G_1 \cup (R_2 \cap R(G_2))/G_2$ where these manifolds are identified along their common boundary $(W \cap R(G))/J$.

Now we consider the Second Combination Theorem. We shall assume that $f \in \mathbf{Mob}_3$, and J_1, J_2 are cyclic loxodromic (or trivial) subgroups of a Kleinian group $G_0 \subset \mathbf{Mob}_3$. Two compact manifolds $B_1, B_2 \subset S^3$ are jointly f -blocked if B_m is (J_m, G_0) -block ($m = 1, 2$), f maps the exterior of B_1 onto the interior of B_2 , and $f \cdot J_1 \cdot f^{-1} = J_2$. If B_1 and B_2 are jointly f -blocked, then let A be equal to $\text{ext}(B_1 \cup B_2)$, $A_0 = S^3 \setminus G_0(B_1 \cup B_2)$.

Examples. Let J be a trivial group, and $B_m \subset R(G_0)$ be contained inside of a fundamental domain of G so that f maps the exterior of B_1 onto the interior of B_2 . Then B_m are jointly f -blocked. For another example consider a compact hyperbolic surface M^2 which has two totally geodesic boundary components E_1, E_2 of equal length. Let $\tilde{M}^2 \subset \mathbb{H}^2$ be the universal cover of M^2 ; then \tilde{E}_j is a component of the preimage of E_j in \mathbb{H}^2 ($j = 1, 2$). Denote by J_j the stabilizer of E_j in $G_0 = \pi_1(M^2) \subset \mathbf{Mob}_3$. Assume that \mathbb{H}^2 is realized as a half-plane in \mathbb{R}^3 , and for $j = 1, 2$ let $B_j \subset \mathbb{R}^3$ be a closed Euclidean ball such that

- (a) $B_j \cap \widetilde{M}^2 = E_j$,
 (b) ∂B_j is orthogonal to \mathbb{H}^2 .

Then there is a Moebius transformation f such that B_1, B_2 are jointly f -blocked.

In this paper we shall always have $B_1 \cap B_2 = \emptyset$.

Theorem 1.2 (Second Maskit Combination Theorem). *Let $J_1, J_2 \subset G_0$, and $f \in \mathbf{Mob}_3$ be as above. Assume that B_1 and B_2 are jointly f -blocked compact submanifolds of \mathbb{S}^3 , and that $A_0 \neq \emptyset$. Let D_0 be a fundamental set for G_0 such that*

- (1) $D_0 \cap B_m$ is a fundamental set for action of J_m on B_m .
 (2) $f(D_0 \cap W_1) = D_0 \cap W_2$ where $W_m = \partial B_m$.

We set $G = \langle G_0, f \rangle$, $D = D_0 \cap (A \cup W_1)$. Then the following statements hold

- (i) $G \cong G_0 *_f$ is the HNN-extension of G_0 by f .
 (ii) G is Kleinian.
 (iii) D is a fundamental set for G .
 (iv) The set A_0 is precisely invariant under G_0 in G .

Let $Q = \text{cl } A_0 \cap R(G_0)$; then $R(G)/G$ is equal to Q/G_0 , where the two boundary components $(W_1 \cap R(G_1))/J_1$ and $(W_2 \cap R(G_2))/J_2$ are identified, this identification being given by f .

Remark 5. We do not formulate combination theorems in the greatest generality, but our formulations suffice for the purposes of this article.

Theorems 1.1 and 1.2 are due to Klein and Maskit. Our formulations repeat [46], who considered only the case $n = 2$, but we drop all two-dimensional assertions of [46, §VII, Theorem C.2, Theorem E.5]. Proofs of Theorems 1.1 and 1.2 follow word by word the proofs of [46, §VII, Theorem C.2, Theorem E.5].

Other generalizations [19], [41, pp. 169–170], [1, Theorem 4.2, 4.5] of combination theorems to higher dimensions also repeat Maskit's original arguments [48].

1.3. 3-Manifolds. We suppose that reader is familiar with basic concepts of three-dimensional topology such as *incompressible surfaces* and *canonical decomposition* of a *Haken* manifold into *hyperbolic* and *Seifert* manifolds (we shall consider the last as total spaces of fiber bundles over 2-dimensional *orbifolds* [56]); see [16], [20], [56], [60, §13] for references.

For construction of finite-sheeted coverings of 3-manifolds we shall frequently use the following results of Hempel [17] and McCullough and Miller [49].

Theorem 1.3. *Let Γ be a finitely generated subgroup of $\text{PSL}(2, \mathbb{C}) = \mathbf{Mob}_2$. Then for all but finite primes $p \in \mathbb{N}$ the group Γ contains a normal*

torsion-free subgroup Γ_0 of finite index such that the intersection of Γ_0 with any maximal parabolic subgroup $P \subset \Gamma$ equals the subgroup $\{\gamma^p : \gamma \in P\}$.

Theorem 1.4 [50, Proposition 4.1]. *Let M be a compact Seifert fiber space over an orbifold O , $p: \tilde{O} \rightarrow O$ be a finite-sheeted covering where \tilde{O} is an orbifold, and p is in the sense of orbifold theory, and $n \in \mathbb{N}$, such that for any component $b \subset \partial \tilde{O}$ the restriction p to b is a n -sheeted covering. Then there exists a compact Seifert fiber space \tilde{M} with the base \tilde{O} and a covering $\tilde{p}: \tilde{M} \rightarrow M$ such that the induced map of bases is $p: \tilde{O} \rightarrow O$, and the regular fiber of \tilde{M} n times covers the regular fiber of M .*

Let the manifold M be obtained by gluing finitely many components M_j by identification of connected incompressible boundary surfaces S_{kj} . Suppose that we have a system of regular finite-sheeted coverings $q_j: \tilde{M}_j \rightarrow M_j$ such that for every S_{kj} and every component $q_j^{-1}(S_{kj})$ the subgroup defining the coverings

$$q_j|_{q_j^{-1}(S_{kj})}: q_j^{-1}(S_{kj}) \rightarrow S_{kj}$$

depends only on S_{kj} and not on the adjacent manifolds.

Theorem 1.5 [17], [50, Proposition 1.1]. *Under the above conditions there exists a finite-sheeted covering $p: \tilde{N} \rightarrow M$ such as the restriction of p to every component \tilde{N}_j covering M_j is equivalent to q_j :*

$$\begin{array}{ccc} \tilde{N} \supset \tilde{N}_j & \rightarrow & \tilde{N}_j \\ \downarrow p & & \downarrow q_j \\ M \supset M_j & \rightarrow & M_j \end{array}$$

Let S be a closed surface, D_1, \dots, D_{2r} pairwise disjoint closed discs in S and $\Sigma = S \setminus (\text{int } D_1 \cup \dots \cup \text{int } D_{2r})$. Then for any positive integer n there exists an n -sheeted ramified cyclic covering $p_0: \tilde{S} \rightarrow S$ such that exactly one branch point of order n lies in every disc D_i (see, e.g. [5]).

Definition 5. Denote the restriction of the covering p_0 to the surface $\tilde{\Sigma} = \Sigma \setminus p^{-1}(\text{int } D_1 \cup \dots \cup \text{int } D_{2r})$ by p . Let M be the product $\Sigma \times \mathbb{S}^1$. Then the covering $\tilde{p}: \tilde{M} \rightarrow M$, that corresponds to p by Theorem 1.4, is the standard n^2 -sheeted covering of the manifold M .

1.4. Flat conformal structures.

Suppose (M, K) is an n -manifold with a flat conformal structure K . Denote by $p: \tilde{M} \rightarrow M$ the universal covering M ; $\pi_1(M)$ is the group of deck transformations of p . Then (M, K) admits a development map $\text{dev}: \tilde{M} \rightarrow \mathbb{S}^n$. This map can be characterized by the following properties:

- (a) dev is conformal with respect to the FCS lifted to \widetilde{M} from (M, K) ,
 (b) $\text{dev} \circ g = \rho(g) \circ \text{dev}$, where ρ is a homomorphism of $\pi_1(M)$ in \mathbf{Mob}_n which is called the *holonomy representation* of (M, K) .

The group $\rho(\pi_1(M))$ is called the *holonomy group* of K .

If (M, K) is uniformizable, then the holonomy $\rho(\pi_1(M))$ is just the uniformizing group G , and $\text{dev} \circ p_\Gamma = p$, where $p_\Gamma: \text{dev}(\widetilde{M}) = D \rightarrow M$ is the uniformization, and Γ is the uniformizing group. In general, development maps can be very complicated (in the worst case they can be onto \mathbb{S}^n). However there are some simple cases where one can guarantee that dev is not so bad. For example, suppose that M is compact, and K is a hyperbolic FCS, i.e., $\text{dev}(\widetilde{M}) \subset \mathbb{H}^n \subset \mathbb{S}^n$ and $\rho(\pi_1(M)) \subset \text{Isom}(\mathbb{H}^n) \subset \mathbf{Mob}_n$; this is equivalent to the assumption that the conformally Euclidean Riemannian metric on M is conformally equivalent to a hyperbolic one. Then the classical theorem of differential geometry implies that the corresponding hyperbolic metric is complete, and $\text{dev}: \widetilde{M} \rightarrow \mathbb{H}^n$ is a diffeomorphism. In particular, the holonomy group $\rho(\pi_1(M)) = G$ is discrete, and G is the *uniformizing group* for (M, K) .

Generalizations of this simple fact can be found in [35], [12], [22], [21], [40]. In particular, if M is compact, and dev is not surjective, then the development map is a covering on its image and (with several simple exceptions) the holonomy group is discrete.

Warning. Even in the this case K can be nonuniformizable. However, (M, K) is “commensurable” with a uniformizable flat conformal manifold via a pair of finite-sheeted coverings.

In this series of articles we have dealt only with uniformizable structures. The following question remains open: Does the existence of a flat conformal structure on a compact manifold imply the existence of an uniformizable one? Probably the answer is “yes” in dimension 3 and “no” in higher dimensions. See [9], [10], [15], [24], [41], [49], [55] for further discussion.

2. Uniformization of Seifert manifolds

2.1. Let M be a Seifert manifold with zero Euler number and hyperbolic base. Then there exist certain $\mathbb{H}^2 \times \mathbb{R}$ -structures on M (see [56]); hence $M = \mathbb{H}^2 \times \mathbb{R} / \Gamma$, where Γ is a torsion-free discrete group of isometries of $\mathbb{H}^2 \times \mathbb{R}$. This group may be chosen so that its cyclic normal subgroup is generated by the displacement $t: (z, \varphi) \rightarrow (z, \varphi + 2\pi)$, where $z \in \mathbb{H}^2$, $\varphi \in \mathbb{R}$. Let $q: \mathbb{H}^2 \times \mathbb{R} \rightarrow \mathbb{H}^2 \times \mathbb{S}^1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3: x_2^2 + x_3^2 > 0$

with the Riemannian metric $ds^2 = |dx|^2 / (x_2^2 + x_3^2)^2$ on $\mathbb{S}^3 \setminus \mathbb{S}^1$ be the universal covering. The deck transformation group of this covering is $\langle t \rangle$. This map induces a homomorphism $q_*: \Gamma \rightarrow \Gamma_0 \subset \mathbf{Mob}_3$. The group Γ_0 acts freely and discontinuously on $\mathbb{H}^2 \times \mathbb{S}^1$, and the manifold $\mathbb{H}^2 \times \mathbb{S}^1 / \Gamma_0$ is homeomorphic to $\mathbb{H}^2 \times \mathbb{R} / \Gamma = M$. So the manifold M admits a flat conformal structure which is uniformized by a ‘‘Fuchsian’’ group Γ . Since the geometries \mathbb{E}^3 and $\mathbb{S}^2 \times \mathbb{R}$ can be realized in \mathbb{R}^3 as $(\mathbb{R}^3, |dx|^2)$ and $(\mathbb{R}^3 \setminus \{0\}, |dx|^2 / |x|^2)$, any Seifert manifold with zero Euler number admits a uniformizable FCS. In contrast to that, any Seifert manifold with nonzero Euler number and Euclidean base orbifold admits no FCS (see [9]).

The main purpose of this section is to prove the following.

Theorem 2.1. *Let $S(g, e)$ be the total space of the circle bundle over the closed orientable surface S_g of genus g , whose Euler number $e \in \mathbb{Z}$ satisfies $0 < e \leq (g - 1)/11$. Then the manifold $S(g, e)$ admits a uniformizable flat conformal structure.*

2.2. We need the following description of the manifold $S(g, e)$. Let $\Sigma_g = S_g \setminus \text{int}(B^2)$, where B^2 is a closed disc, $x \in \partial B^2$, $\mathcal{N} = \Sigma_g \times \mathbb{S}^1$, $t = \{x\} \times \mathbb{S}^1 \subset \partial \mathcal{N}$, and $\beta = \partial B^2 \times \{\varphi\}$, where $\varphi \in \mathbb{S}^1$ and $T = \partial B^2 \times \mathbb{S}^1$ is the boundary of \mathcal{N} . Let $\mathcal{T} = B^2 \times \mathbb{S}^1$ be a solid torus, $\tau = \{x\} \times \mathbb{S}^1 \subset \partial \mathcal{T}$, and $\kappa = \partial B^2 \times \{\varphi\} \subset \partial \mathcal{T}$. We shall denote the corresponding elements of $\pi_1(T)$ and $\pi_1(\mathcal{T})$ by the same symbols: t, β, τ, κ . The manifold $S(g, e)$ is obtained by gluing of \mathcal{N} and \mathcal{T} so that the loop t is identified with τ and the loop β is identified with $\kappa \cdot t^e$.

2.3. Proof of Theorem 2.1. Our main purpose is to construct a Kleinian group $H = H(g, 1)$ such that $R(H)/H = M(H)$ is homeomorphic to $S(g, 1)$, where $g = 12$. A fundamental polyhedron Φ for the action of H on $R(H)$ is homeomorphic to a solid torus and satisfies the following properties

(a) Faces $Q_1, R'_1, Q'_1, R_1, \dots, Q_g, R'_g, Q'_g, R_g, Q_1$ of Φ are Moebius equivalent to standard annuli. Two consecutive faces in this list intersect each other in Euclidean circles; all other pairs of faces have empty intersection (see Figure 2, next page).

The faces of Φ are paired by Moebius transformations $A_1: Q_1 \rightarrow Q'_1, B_1: R_1 \rightarrow R'_1, \dots, A_g: Q_g \rightarrow Q'_g, B_g: R_g \rightarrow R'_g$ which generate H . Let x_0 be a point of the circle $Q_1 \cap R_g$,

$$x_1 = B_1^{-1} \circ A_1^{-1} \circ B_1 \circ A_1(x_0) = [A_1, B_1](x_0) \in Q_2 \cap R_1,$$

and so on,

$$x_g = [A_g, B_g] \circ \dots \circ [A_1, B_1](x_0) \in R_g \cap Q_1.$$

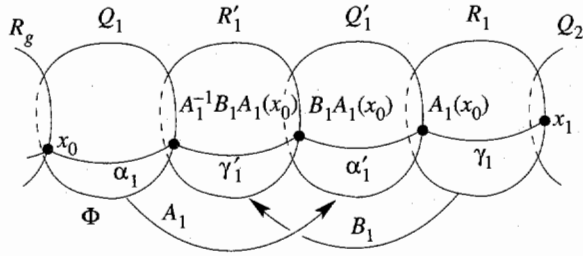


FIGURE 2

(b) We require x_g to equal x_0 and the sum of dihedral angles of the polyhedron Φ to equal 2π . Then Φ is a fundamental domain for the group

$$H = \langle A_1, B_1, \dots, A_g, B_g : [A_g, B_g] \cdots [A_1, B_1] = 1 \rangle.$$

To see this it is sufficient to continue the polyhedron Φ to the hyperbolic space $\mathbb{H}^4 = \mathbb{R}_+^4 = \{(x_1, x_2, x_3, x_4) : x_4 > 0\}$ (each sphere is continued to a geodesic hyperplane) and apply the Poincaré theorem on fundamental polyhedra [46].

Let α_1 be a simple closed curve on Q_1 which connects points x_0 and $A_1^{-1} \cdot B_1 \cdot A_1(x_0)$, and let curve $\gamma_1 \subset R_1$ connect the point $A_1(x_0)$ with x_1 , $\alpha'_1 = A_1(\alpha_1)$, and $\gamma'_1 = B_1(\gamma_1)$ (see Figure 2). By analogy we construct the curves $\alpha_2, \alpha'_2, \gamma_2, \gamma'_2, \dots, \alpha_g, \alpha'_g, \gamma_g, \gamma'_g$. Their union η is a simple closed curve on $\partial\Phi$.

(c) Suppose that the linking number e of the curve η and the axis of the solid torus $\mathbb{S}^3 \setminus \Phi$ equals 1. It is easy to see that this condition is equivalent to the following one: the loop η is homotopic on $\partial\Phi$ to the loop $t+k$, where $t = Q_1 \cap R_g$, and the class $[k]$ generates the kernel of $\pi_1(\partial\Phi) \rightarrow \pi_1(\Phi)$, under appropriate choices of orientation on the above loops.

2.4. Now we show that conditions (a)–(c) suffice for H to uniformize the manifold $S(g, 1)$. Let $T' \subset \Phi$ be a torus which is parallel to $\partial\Phi$, and \mathcal{F} be the component of $\Phi \setminus T'$ lying between $\partial\Phi$ and T' . The manifold $M(H) = R(H)/H$ is homeomorphic to Φ/H . Let $q: \Phi \rightarrow M(H)$ be the natural projection, $\mathcal{N} = q(\mathcal{F})$, $\beta = q(\beta')$ where β' is a loop on T' parallel to η in $\Phi \setminus \mathcal{F}$. The manifold \mathcal{N} is homeomorphic to $\Sigma_g \times \mathbb{S}^1$, and the manifold $M(H)$ is obtained by gluing \mathcal{N} and the solid torus $\mathcal{T} = q(\Phi \setminus \mathcal{F})$ essentially in the same way as in §2.2, where we put $|e| = 1$. Therefore we have $M(H) = S(g, 1)$.

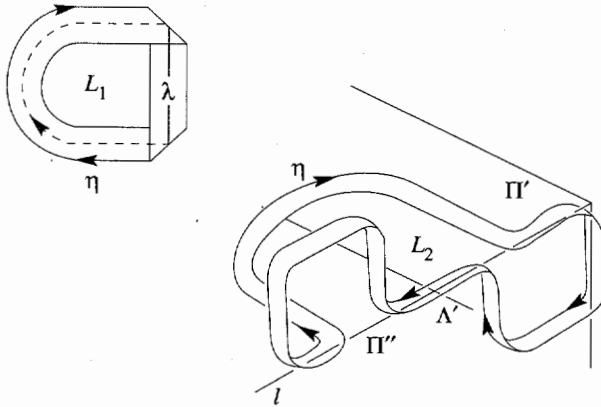


FIGURE 3

2.5. Construction of the polyhedron Φ for $g = 12, e = 1$. Notice that on the twisted strip L_1 (Figure 3) the linking number of the boundary curve η and the “middle line” λ equals 1. In Figure 3 the strip L_2 is drawn so that it is equivalent to L_1 and has no “overlaps”. Our aim is to cover L_2 by spheres to satisfy conditions (a)–(c) of §2.3. Let Π', Π'' be the horizontal and vertical planes respectively (see Figure 3).

We single out two parts of the strip L_2 : the part L'_2 , which is the connected component of $L_2 \cap \Pi'$ not intersecting Π'' , and the part $L''_2 = L_2 \setminus L'_2$. Let $l = \Pi' \cap \Pi''$, and let $\Lambda' \subset \Pi''$ be the axis of symmetry of the substrip L''_2 , and $O = l \cap \Lambda'$. We consider l and Λ' as the coordinate axes in the plane Π' .

Let O_1 and O_2 be points on the plane Π' with coordinates $(0, 1)$ and $(2, 1)$ respectively; then $l_1 \subset \Pi'$ is a straight line passing through the points O_1 and O_2 . Next we put $\alpha = \pi/8, \varepsilon = \pi/24$ and the point $\mathcal{E}_1 \in \Pi'$ having the coordinates $(1, 1 - \tan(\alpha/2))$. Let O_j^\perp be the straight line orthogonal to Π' and containing O_j . We choose Q_1 to be the sphere with center C_1 and radius $r = \tan(\alpha/2)/\cos(\varepsilon/2)$ (the same letter Q_1 will denote the face of the polyhedron Φ that lies on this sphere). Spheres R'_1, Q'_1, R_1 , and Q_2 arise by rotating the sphere Q_1 around the axis O_2^\perp with the angles $\alpha, 2\alpha, 3\alpha, 4\alpha$. By analogy, the spheres R_{12}, Q'_{12}, R'_{12} , and Q_{12} arise by rotating Q_1 around the axis O_1^\perp with the same angles (see Figure 4, next page). It is easy to see that the angles between the neighboring spheres equal ε and the centers of R_1 and Q_1 lie on the axis l . In this way we have “covered” the strip L' .

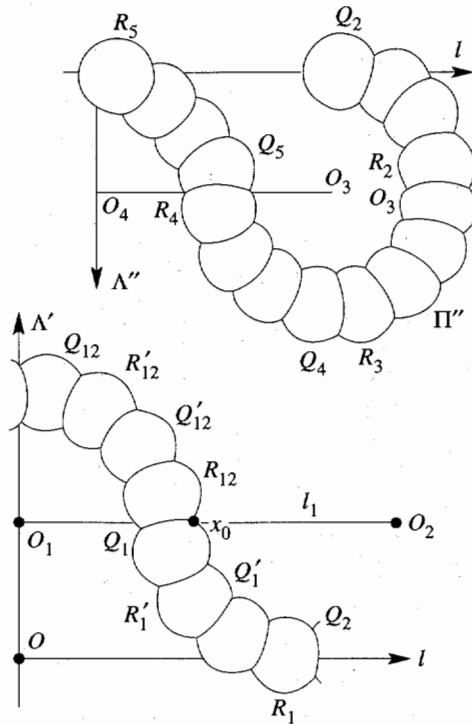


FIGURE 4

Let J_1 be the inversion in the sphere Q_1 , and σ_1 be the symmetry in the plane that passes through O_2^\perp and the center of R'_1 ; then we put $A_1 = \sigma_1 \circ J_1$. Similarly, let I_1 be an inversion in the sphere R_1 , θ_1 be a symmetry in the plane that passes through O_2^\perp and the center of Q'_1 , $B_1 = \theta_1 \circ I_1$. It is easy to see that $A_1(Q_1) = Q'_1$, $B_1(R_1) = R'_1$, $A_1(Q_1 \cap R'_1) = R'_1 \cap Q'_1$ and so on.

Now we consider the strip L'' . Let $\Lambda'' \subset \Pi''$ be a straight line orthogonal to l and passing through the point O . We shall consider (l, Λ'') as coordinate axes on Π'' (Figure 4); l is directed to the "right" and Λ'' has direction "down". Let $O_3 = (2, 1)$, $O_4 = (0, 1)$ be the points on the plane Π'' , and Q_3^\perp , Q_4^\perp straight lines passing through Q_3 , Q_4 orthogonally to Π'' . Then the spheres R'_2 , Q'_2 , R_2, \dots, R_4 , Q_5 arise by rotating Q_2 around Q_3^\perp with angles $\alpha, 2\alpha, 3\alpha, \dots, 11\alpha, 12\alpha$. All these spheres are orthogonal to Π'' and have angles of intersection equal to ε . Finally, the spheres R'_5 , Q'_5 , and R_5 arise by rotating Q_5 around O_4^\perp with angles $\alpha, 2\alpha, 3\alpha$. The center of the sphere R_5 lies on the line l .

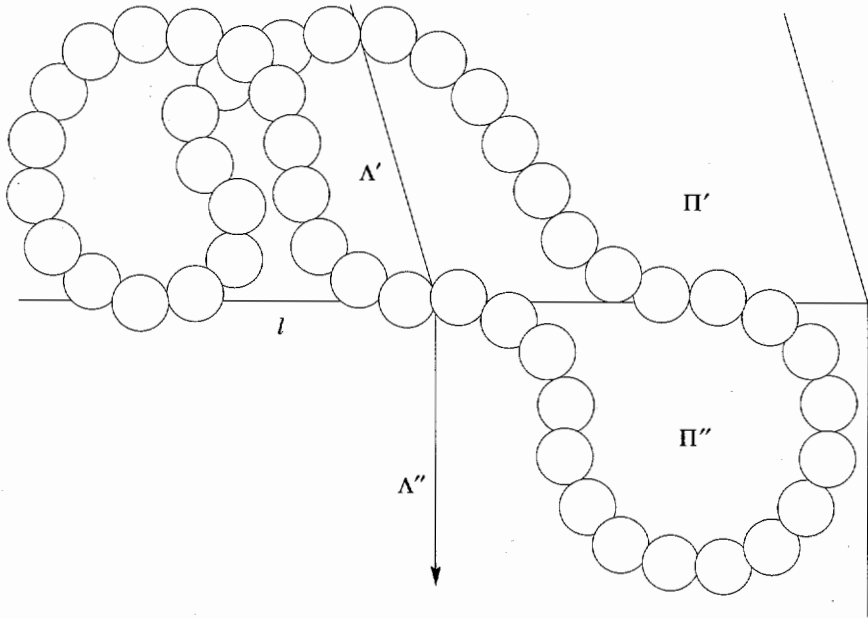


FIGURE 5

The system of spheres $Q_6, R'_6, \dots, Q'_{11}, R_{11}$ is obtained from the family $Q_{12}, R'_{12}, \dots, Q'_5, R_5$ via the symmetry in the line Λ' . The angle between any two neighboring spheres equals ε . The intersection $\text{ext}(Q_1) \cap \dots \cap \text{ext}(R_{12})$ is precisely the polyhedron Φ we were looking for. The final configuration is given in Figure 5.

The sum of dihedral angles is equal to $48\varepsilon = 2\pi$. The generators $A_2, B_2, \dots, A_{12}, B_{12}$ may be chosen in the same way as A_1 and B_1 : $A_i = \sigma_i \circ J_i, B_i = \theta_i \circ I_i$ where J_i and I_i are inversions in Q_i and R_i ; the transformations σ_i and θ_i are symmetries in the Euclidean perpendicular bisectors of the lines joining centers of Q_i, Q'_i and R_i, R'_i respectively.

Let $x_0 \in Q_1 \cap l_1$ be the point closest to O_2 . It is easy to see that

$$[A_{12}, B_{12}] \circ \dots \circ [A_1, B_1](x_0) = x_0,$$

and the curve η on $\partial\Phi$, constructed according to §2.3, has the linking number 1 with the axis λ of the solid torus $S^3 \setminus \Phi$.

Thus, the group $H = H(12, 1)$ is constructed.

2.6. Here we will show that for any g and e such that $1 \leq |e| \leq (g-1)/11$, there exists a group $H(g, e)$ uniformizing the manifold $S(g, e)$. Let H be a subgroup in the group $H(12, 1)$ of the index j . Then we

have $H = H(11j + 1, j)$ by Lemma 3.5 of [56] and the Riemann-Hurwitz formula. Therefore, for any given $e = j > 0$ we have constructed a group $H(g, e)$ with $g = 11e + 1$ or equivalently $e = (g - 1)/11$. So to complete the proof of Theorem 2.1 we only have to construct the group $H(g, e)$ with $g = 11e + k$ for any $k > 0$.

Denote by Π the Euclidean plane that passes through the line l_1 orthogonally to Π' , and let B be that component of $\mathbb{R}^3 \setminus \Pi$ which contains the sphere Q_{12} . Next we put $\bar{\Pi} = \Pi \cup \{\infty\}$ and $\bar{B} = \text{cl}(B) \cup \{\infty\}$. Denote by h the hyperbolic transformation $[A_{10}, B_{10}] \circ \cdots \circ [A_1, B_1]$. The fixed point set of h is the intersection of the straight line l_1 and the circle $C \subset \Pi'$ with center O_1 and radius $1 - r^2 \sin^2(\varepsilon/2)$. It is easy to see that the sphere $\bar{\Pi}$ is precisely invariant in the group $H(12, 1)$ with respect to $\langle h \rangle$.

We can choose a subgroup H of any prescribed index e in $H(12, 1)$ such that $H \supset \langle A_{11}, B_{11}, A_{12}, B_{12} \rangle$. So the group H is the Maskit combination of the groups $\langle A_{11}, B_{11}, A_{12}, B_{12} \rangle$ and $G(11e - 1, e)$.

To construct $H(11e + 1 + k, e)$ for any $k > 0$ it is sufficient to replace the subgroup $\langle A_{11}, B_{11}, A_{12}, B_{12} \rangle$ by a free Fuchsian group $F_{2(2+k)}$ of rank $2(2 + k)$ such that

- (1) the circle C is invariant under the action of this group,
- (2) $\langle h \rangle \subset [F_{2(2+k)}, F_{2(2+k)}]$,
- (3) the ball $\mathbb{R}^3 \setminus B$ is precisely invariant in $F_{2(2+k)}$ with respect to $\langle h \rangle$.

The groups $F_{(2+k)k}$ and $G(11e - 1, e)$ satisfy the conditions of the Maskit Combination Theorem (Theorem 1.1) with the amalgamated subgroup $\langle h \rangle$. Therefore the group $\langle F_{2(2+k)}, G(11e - 1, e) \rangle$ uniformizes the manifold $S(11e - 1 + 2 + k, e) = S(11e + 1 + k, e)$ which is obtained by gluing $S^1 \times \Sigma_{2+k}$ and $S^1 \times \Sigma_{11e-1}$. For more details see [30, 3.2–3.4].

So Theorem 2.1 is proved.

2.7. Let $\tilde{H}(g, e)$ be the extension of the group $H(g, e)$ into the space \mathbb{H}^4 ; the manifold $\mathbb{H}^4/\tilde{H}(g, e)$ is homeomorphic to the 2-plane bundle over S_g with Euler number e . This may be seen as follows. Choose a fundamental polyhedron Φ for $H(g, e)$ such that $\partial\Phi$ consists of annuli lying on Euclidean spheres (cf. §2.3). The convex hull $\tilde{\Phi}$ of Φ in \mathbb{H}^4 is a fundamental polyhedron for the action of $\tilde{H}(g, e)$ in \mathbb{H}^4 . The polyhedron $\tilde{\Phi}$ admits a natural \mathbb{R}^2 -fibration which is invariant under the action of $\tilde{H}(g, e)$. This fibration projects to a fibration of

$$M(H(g, e)) = \mathbb{H}^4 \cup R(H(g, e))/\tilde{H}(g, e),$$

whose restriction to $\partial M(H(g, e))$ is a circle fibration over $S(g, e)$.

So for any g and e such that $0 < e \leq (g - 1)/11$ the total space $E(g, e)$ of 2-plane fiber bundle over S_g with the Euler number e admits a complete hyperbolic structure.

Remark 6. As shown by Kuiper [36], [37], the condition $0 < e \leq 2(g - 1)/3$ is sufficient for existence of FCS on $S(g, e)$ and complete hyperbolic structure on $E(g, e)$ (see also [44]) however we will not go into details.

Corollary 2.1. *Any Seifert fiber space with hyperbolic base is virtually conformally flat, i.e., has a finite-sheeted covering space which admits a FCS.*

Proof. It is sufficient to consider only orientable Seifert manifolds M with nonzero Euler numbers. Consider the short exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \xrightarrow{\varphi} F \rightarrow 1$$

where F is a discrete subgroup of $\text{Isom}(\mathbb{H}^2)$. Hence the group F contains a finite-index subgroup F_0 isomorphic to $\pi_1(S_g)$ where $g \geq 12$. The group $G_0 = \varphi^{-1}(F_0)$ has the presentation

$$\langle a_1, b_1, \dots, a_g, b_g, t: [a_i, t] = [b_j, t] = [a_1, b_1] \cdots [a_g, b_g] \cdot t^{-e} = 1 \rangle,$$

where $e \neq 0$. If we put $\tau = t^e$, then the subgroup

$$G'_0 = \langle a_1, b_1, \dots, a_g, b_g, \tau: [a_1, b_1] \cdots [a_g, b_g] = \tau \rangle$$

has a finite index in $\pi_1(M)$ and defines a covering $M_0 \rightarrow M$ such that M_0 admits a flat conformal structure (by Theorem 2.1).

2.8. Application to quasiconformal groups. We recall that a group Γ of homeomorphisms acting on \mathbb{S}^n is said to be (*uniformly*) *quasiconformal* if there exists a number $K < \infty$ such that each element $\gamma \in \Gamma$ is K -quasiconformal map (see [62], [44]).

If $n = 2$, then every quasiconformal group is topologically conjugate to some group of Moebius transformations [61]. The papers [61], [62], [7] give examples disproving the conjecture that the analogous statement is true for $n > 2$. Articles [7] and [44] provide discrete examples of such groups.

Below we show how to construct an analogous example of action of the group $\mathbb{Z}_n \times \pi_1(S_g)$ on \mathbb{S}^3 . Let $H = H(12, 1)$ be the group constructed in Theorem 2.1. Let $\varphi: M(H) \rightarrow M(H)$ be an order- n diffeomorphism isotopic to the identity (it exists due to the \mathbb{S}^1 -action on $M(H) = S(12, 1)$). This diffeomorphism admits a lift $\tilde{\varphi}: R(H) \rightarrow R(H)$ of order n . The restriction of $\tilde{\varphi}$ to the compact fundamental domain Φ of the group H is

smooth and hence is K -quasiconformal for some $K > 1$. For any $h \in H$ we have $h \circ \tilde{\varphi} = \tilde{\varphi} \circ h$; therefore the map $\tilde{\varphi}$ is K -quasiconformal itself. It is sufficient to repeat the considerations of Maskit [46] to prove that $\tilde{\varphi}$ admits a homeomorphic continuation f to the sphere \mathbb{S}^3 .

Furthermore, considerations of Bers [4, Lemma 2] imply that the map f is K -quasiconformal. The group $\Gamma = \langle H, f \rangle$ is isomorphic to $\mathbb{Z}_n \times \pi_1(S_g)$ and defines a K -quasiconformal action on \mathbb{S}^3 . We may apply the above reasoning to construct an \mathbb{S}^1 -action on \mathbb{S}^3 , which is H -equivariant and $L(H)$ is the fixed-point set for this \mathbb{S}^1 -action. Hence the homeomorphism f is topologically conjugate to some Euclidean rotation (see [54]), and $L(H)$ is a tame unknotted topological circle in \mathbb{S}^3 . Any element of $\Gamma \setminus \langle f \rangle$ is "hyperbolic" in the sense of [8], and hence is topologically conjugate to some Moebius transformation. Consequently any element of Γ is conformal up to topological conjugation; however the following statement holds.

Corollary 2.2 [23]. *The group Γ is not topologically conjugate to any subgroup of \mathbf{Mob}_3 .*

Proof. Suppose that such a conjugation g exists; then under the action of the group $G = g \cdot \Gamma \cdot g^{-1} \subset \mathbf{Mob}_3$ the Euclidean circle $\text{Fix}(g \cdot f \cdot g^{-1})$ is invariant. The manifold $M(G) = R(G)/G$ is homeomorphic to $M(\Gamma) = R(\Gamma)/\Gamma$ and has a nonzero Euler number. However this contradicts the existence of an $\mathbb{H}^2 \times \mathbb{R}$ structure on the manifold $M(g \cdot H \cdot g^{-1})$ (see §2.1). q.e.d.

For another interesting example of quasiconformal group see [31, 6.5].

Using similar techniques as in the proof of Corollary 2.2, we also obtain:

Corollary 2.3. *Let M be a closed Seifert manifold with a hyperbolic base. Let Γ be a Kleinian group such that $M = \Omega/\Gamma$, where Ω is an invariant component of $R(\Gamma)$, and Γ acts freely on Ω . Then $\Omega = R(\Gamma)$, and the limit set $L(\Gamma) = \mathbb{S}^3 \setminus \Omega$ is a tame unknotted topological circle.*

See [22] for the case of zero Euler number. This corollary answers a question of Kuiper [36].

2.10. Flat conformal structures on manifolds $S(g, e)$, $e \neq 0$ provide us another interesting example of pathology—disconnectedness of the moduli space $C(M)$ of all FCS on the manifold $M = S(g, e)$. Definitions of topology on this space may be found in [43], [6]. Let $\nu(e, g)$ denote the greatest integer $[g - 1/11e]$.

Theorem 2.2. *Let M be a manifold $S(g, e)$. Then the space $C(M)$ consists of at least $\nu(e, g)$ connected components.*

We only sketch the proof of this theorem since a detailed proof would lead us too far from the main subject of this paper. For the complete

proof see [26]. Below we indicate $\nu(g, e)$ structures on M , which lie in different components of $C(M)$.

Consider the set of manifolds $\mathcal{E} = \{S(n \cdot e, g) : 0 < n \leq \nu(e, g)\}$. All manifolds of \mathcal{E} admit uniformizable FCS K_n , by Theorem 2.1. It is easy to see that there exists a covering $p: S(g, e) \rightarrow S(g, e \cdot n)$, and hence the structures K_n lift to structures \tilde{K}_n on the manifold $S(g, e)$. Then the holonomy groups of the structures \tilde{K}_n are the groups $H(g, n \cdot e)$. The groups $H(g, m \cdot e)$ and $H(g, n \cdot e)$ cannot be deformed one to other in the space of all pseudofuchsian groups (if $n \neq m$). Therefore, results of [22], [23] imply that the structures \tilde{K}_n and \tilde{K}_m lie in different components of $C(M)$. Another way to prove this statement is to distinguish the connected components containing \tilde{K}_n and \tilde{K}_m by the η -invariants associated with conformally Euclidean metrics [26].

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References

- [1] B. Apanasov, *Discrete groups and structures of manifolds*, Novosibirsk, 1983 (Russian).
- [2] B. Apanasov & A. Tetenov, *Nontrivial cobordisms with geometrically finite hyperbolic structures*, J. Differential Geometry **28** (1988) 407–422.
- [3] A. F. Beardon, *The geometry of discrete groups*, Springer, Berlin, 1983.
- [4] L. Bers, *On moduli of Kleinian groups*, Russian Math. Surveys **29** (1974) 86–102.
- [5] A. Edmonds J. Ewing & P. R. Kulkarni, *Regular tessellations of surfaces and $(p, q, 2)$ -triangle groups*, Ann. of Math. (2) **116** (1982) 113–132.
- [6] D. B. A. Epstein, R. Canary & P. Green, *Notes on notes of Thurston*, London Math. Soc. Lecture Notes **111** (1987) 3–92.
- [7] M. Freedman & R. Skora, *Strange actions of groups on spheres*, J. Differential Geometry **25** (1987) 75–98.
- [8] F. Gehring & G. Martin, *Convergence groups*, Proc. London Math. Soc. (3) **55** (1987) 331–358.
- [9] W. Goldman, *Conformally flat manifolds with nilpotent holonomy*, Trans. Amer. Math. Soc. **278** (1983) 573–583.
- [10] —, *Projective structures with Fuchsian holonomy*, J. Differential Geometry **25** (1987) 297–326.
- [11] N. Gusevskii, *On fake Schottky groups*, preprint, Novosibirsk, 1991.
- [12] N. Gusevskii & M. Kapovich, *Conformal structures on 3-manifolds*, Soviet Math. Dokl. **34** (1987) 314–318.

- [13] N. Gusevskii & E. Klimenko, *Fake Schottky groups acting on 3-sphere*, preprint, Novosibirsk, 1991.
- [14] M. Gromov, H. B. Lawson & W. Thurston, *Hyperbolic 4-manifolds and conformally flat 3-manifolds*, Inst. Hautes Études Sci. Publ. Math. **68** (1988).
- [15] M. Gromov & W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987) 1–12.
- [16] J. Hempel, *3-manifolds*, Princeton University Press, Princeton, NJ, 1976.
- [17] —, *Residual finiteness for 3-manifolds*, Annals of Math. Studies **111** (1987), 373–396.
- [18] N. Isachenko, *On uniformly quasiconformal groups that are not isomorphic to Moebius groups*, Soviet Math. Dokl. **42** (1991), No. 1, 125–128.
- [19] D. Ivascu, *On Klein-Maskit combination theorem*, Lecture Notes in Math., Vol. 743, Springer, Berlin, 1976, 115–124.
- [20] W. Jaco & P. Shalen, *Seifert fibre spaces in 3-manifolds*, Mem. Amer. Math. Soc. **2** (1979).
- [21] Y. Kamishima, *Conformally flat manifolds whose development maps are not surjective*, Trans. Amer. Math. Soc. **294** (1986) 607–621.
- [22] M. E. Kapovich, *Some properties of developments of conformal structures on 3-manifolds*, Soviet. Math. Dokl. **35** (1987) 146–149.
- [23] —, *Flat conformal structures on 3-manifolds*, Siberian Branch of USSR Academy of Sciences, Institute of Mathematics, 1987, preprint no. 17 (in Russian).
- [24] —, *On conformal structures with Fuchsian holonomy*, Soviet Math. Dokl., **38** (1989), No. 1, 14–17.
- [25] —, *Flat conformal structures on 3-manifolds. I*, Siberian Math. J. **30** (1989), No. 5, 60–73.
- [26] —, *Deformation spaces of flat conformal structures*, Questions and Answers, General Topology **8** (1990) 253–264.
- [27] —, *Deformations of representations of discrete subgroup of $SO(3,1)$* , preprint, IHES, 1990.
- [28] —, *Intersection pairing on hyperbolic 4-manifolds*, preprint, MSRI, 1992.
- [29] —, *Topological aspects of Kleinian groups in several dimensions*, preprint MSRI, 1992.
- [30] —, *Flat conformal structures on 3-manifolds. II: Uniformization of open hyperbolic and Seifert manifolds*, J. Differential Geometry, to appear.
- [31] —, *Flat conformal structures on 3-manifolds. III: Conformal gluing of hyperbolic and Seifert manifolds*, J. Differential Geometry, to appear.
- [32] S. Kojima, *Isometry groups of hyperbolic 3-manifolds*, Topology Appl. **29** (1988) 297–307.
- [33] S. Kojima, K. Ohshika & T. Soma, *Towards a proof of Thurston's geometrization conjecture for orbifolds*, RIMS, No. 568, 1985, 1–72, preprint.
- [34] N. Kuiper, *On conformally-flat spaces in the large*, Ann. of Math. (2) **50** (1949) 916–924.
- [35] —, *On compact conformally Euclidean spaces of dimension > 2* , Ann. of Math. (2) **52** (1950) 478–490.
- [36] —, *Hyperbolic 4-manifolds and tessellations*, Publ. Math. Inst. Hautes Études Sci. **68** (1988).
- [37] —, *Fairly symmetric hyperbolic manifolds*, Geometry and Topology of Submanifolds II, World Sci. Publ., Singapore, 1990, 165–203.
- [38] R. Kulkarni, *On principle of uniformization*, J. Differential Geometry **13** (1978) 109–138.
- [39] —, *Conformal structures and Moebius structures*, Conformal Geometry, Aspects of Math., Vol. 12 (R. Kulkarni & U. Pinkall, eds.), Friedr. Vieweg, Braunschweig and Wiesbaden, 1988, 1–39.

- [40] R. Kulkarni & U. Pinkall, *Uniformizations of geometric structures with applications to conformal geometry*, Lecture Notes in Math., Vol. 1209, Springer, Berlin, 1986, 190–209.
- [41] ———, *A canonical metric for Moebius structures and its applications*, preprint of Mittag-Leffler Inst., 1990.
- [42] S. L. Krushkal', B. N. Apanasov & N. A. Gusevskii, *Kleinian groups and uniformization in examples and problems*, Transl. Math. Monographs, Vol. 62, Amer. Math. Soc., Providence, RI, 1986.
- [43] L. Lok, *Deformations of locally homogeneous spaces*, Thesis, Columbia University, 1984.
- [44] F. Luo, *Constructing flat conformal structures on some Seifert manifolds*, Math. Ann. **294** (1992) 449–458.
- [45] G. Martin, *Discrete quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **11** (1986) 179–202.
- [46] B. Maskit, *Kleinian groups*, Springer, Heidelberg, 1987.
- [47] ———, *Self-maps of Kleinian groups*, Amer. J. Math. **93** (1971) 840–856.
- [48] ———, *On Klein combination theorem. III*, Advances in the Theory of Riemann Surfaces, Princeton University Press, Princeton, NJ, 1971.
- [49] S. Matsumoto, *Foundations of flat conformal structure*, preprint, 1990.
- [50] D. McCullough & A. Miller, *Manifold covers of 3-orbifolds with geometric pieces*, Topology Appl. **31** (1989) 169–185.
- [51] C. McMullen, *Iterations on Teichmuller space*, Invent. Math. **99** (1989) 425–454.
- [52] W. Meeks & P. Scott, *Finite group actions on 3-manifolds*, Invent. Math. **86** (1986) 287–346.
- [53] J. Morgan, *On Thurston's uniformization theorem for 3-dimensional manifolds*, Smith Conjecture, Academic Press, New York, 1984.
- [54] F. Raymond, *Classification of actions of the circle*, Trans. Amer. Math. Soc. **131** (1968) 51–78.
- [55] R. Schoen & S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988) 47–71.
- [56] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983) 401–478.
- [57] W. Thurston, *Hyperbolic structures on 3-manifolds. I*, Ann. of Math. (2) **124** (1986) 203–246.
- [58] ———, *Hyperbolic structures on 3-manifolds. II: Surface groups and 3-manifolds which fiber over the circle*, Ann. of Math., to appear.
- [59] ———, *Hyperbolic structures on 3-manifolds. III: Deformations of 3-manifolds with incompressible boundary*, Ann. of Math., to appear.
- [60] ———, *Geometry and topology of 3-manifolds*, Princeton University Lecture Notes, 1979–81.
- [61] P. Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980) 73–78.
- [62] ———, *A quasiconformal group not isomorphic to Moebius*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981) 149–160.
- [63] J. Vaisala, *Lectures on n -dimensional quasiconformal mappings*, Lecture Notes in Math., Vol. 229, Springer, Berlin, 1971.